

# Polyconvexity and existence theorem for nonlinearly elastic shells

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## Abstract

We present an existence theorem for a large class of nonlinearly elastic shells. We restrict our discussion to hyperelastic materials, that is to elastic materials possessing a stored energy function. We define the notion of a polyconvex and orientation preserving stored energy function for shells and we give an example of such a function.

## 1 Introduction

A shell is a three-dimensional continuum which occupies a volume contained between two surfaces (in general parallel) close to each other. A natural way to define a shell is to consider a surface  $S$  embedded in  $\mathbb{R}^3$  and to thicken it on each side. For non-extreme aspect to thickness ratios, one can use 3D FEM codes but for ultrathin materials, such as thin polymeric films or biological membranes, a 2D shell model is needed.

In the mathematical analysis of 3D hyperelasticity, the lack of convexity of the stored energy function stood for a long while as a major difficulty in establishing an existence theorem until J. M. Ball was able to overcome it in a landmark paper (Ball, 1977) by means of the weaker requirement of polyconvexity.

The purpose of this paper, inspired by the approach of J. M. Ball, is to present a general theorem of existence of global minimizers to a large class of nonlinear shell models under realistic hypothesis.

To give the starting point of our result, let us briefly recall the framework considered in the context of three-dimensional elasticity. Let  $\Omega \subset \mathbb{R}^3$  be a

domain considered as the reference configuration of an elastic body. The admissible deformations  $\Theta : \Omega \rightarrow \mathbb{R}^3$  satisfy

$$\det \nabla \Theta > 0.$$

Now we consider a shell  $\mathcal{C}$  with thickness  $2\varepsilon > 0$  whose reference configuration is the set

$$\mathcal{C} = \{\Phi(x, z) = \varphi(x) + z\mathbf{a}_3(x), \quad (x, z) \in \Omega := \omega \times (-\varepsilon, \varepsilon)\}$$

where  $\omega \subset \mathbb{R}^2$  is a domain and

$$\mathbf{a}_3(x) := \frac{\partial_1 \varphi(x) \wedge \partial_2 \varphi(x)}{|\partial_1 \varphi(x) \wedge \partial_2 \varphi(x)|}$$

is the unit normal vector to the middle surface  $S := \varphi(\omega)$ . We make the realistic assumption that the deformations  $\Theta : \mathcal{C} \rightarrow \mathbb{R}^3$  of the shell are of the form

$$\Theta(\Phi(x, z)) = \psi(x) + z\mathbf{a}_3(\psi)(x), \quad (x, z) \in \Omega,$$

where

$$\mathbf{a}_3(\psi)(x) := \frac{\partial_1 \psi(x) \wedge \partial_2 \psi(x)}{|\partial_1 \psi(x) \wedge \partial_2 \psi(x)|}$$

is the unit normal vector to the deformed middle surface  $\hat{S} := \psi(\omega)$  of the shell. By letting

$$\Psi(x, z) := \Theta \circ \Phi(x, z) = \psi(x) + z\mathbf{a}_3(\psi)(x),$$

it follows that

$$\det \nabla \Psi(x, z) = \det \nabla \Theta(\Phi(x, z)) \det \nabla \Phi(x, z)$$

Hence, in order that the condition  $\det \nabla \Theta(\Phi(x, z)) > 0$  may be satisfied, we will impose

$$\det \nabla \Phi > 0 \quad \text{and} \quad \det \nabla \Psi > 0.$$

Thus, since

$$\det \nabla \Psi = \left(1 - \frac{z}{R_1(\psi)}\right) \left(1 - \frac{z}{R_2(\psi)}\right) |\partial_1 \psi \wedge \partial_2 \psi|$$

where  $1/R_1(\boldsymbol{\psi})$ ,  $1/R_2(\boldsymbol{\psi})$  are the principal curvatures of the deformed middle surface, we impose

$$\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi} \neq \mathbf{0} \quad \text{and} \quad \max_{\alpha \in \{1,2\}} \left| \frac{\varepsilon}{R_\alpha(\boldsymbol{\psi})} \right| < 1.$$

Note that the condition  $\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi} \neq \mathbf{0}$  is already mentioned in (Ciarlet et al., 2013) where the authors present a notion of polyconvexity and orientation-preserving condition for a surface. However, for a shell, this condition is not sufficient to ensure the local injectivity of the deformation.

## 2 Preliminaries

In all that follows, Greek indices and exponents range in the set  $\{1, 2\}$  while Latin indices and exponents range in the set  $\{1, 2, 3\}$  (save when they are used for indexing sequences). We use the Einstein summation convention with respect to repeated indices and exponents.

The three-dimensional Euclidian space is identified with  $\mathbb{R}^3$  by choosing an origin and a Euclidian basis. Vector and tensor fields are denoted by boldface letters. The Euclidian norm, the inner product, the vector product and the tensor product of two vectors  $\boldsymbol{u}$  and  $\boldsymbol{v}$  in  $\mathbb{R}^3$  are respectively denoted  $|\boldsymbol{u}|$ ,  $\boldsymbol{u} \cdot \boldsymbol{v}$ ,  $\boldsymbol{u} \wedge \boldsymbol{v}$  and  $\boldsymbol{u} \otimes \boldsymbol{v}$ . The sets of all  $n \times n$  real matrices and of all  $m \times n$  real matrices are respectively denoted  $\mathbb{M}^n$  and  $\mathbb{M}^{m \times n}$ . The notation  $|\boldsymbol{A}|$  designate the Frobenius norm of a real matrix  $\boldsymbol{A} \in \mathbb{M}^{m \times n}$  defined by  $|\boldsymbol{A}| := \text{tr}(\boldsymbol{A}^T \boldsymbol{A})^{1/2}$ .

A domain  $\omega \subset \mathbb{R}^2$  is a bounded, connected, open set with a Lipschitz-continuous boundary  $\gamma := \partial\omega$ , the set  $\omega$  being locally on the same side of  $\gamma$ . A generic point in the set  $\overline{\omega}$  is denoted by  $x = (x_\alpha)$  and partial derivatives, in the classical or distributional sense, are denoted  $\partial_\alpha := \partial/\partial x_\alpha$ .

The notation  $L^p(\omega; \mathbb{R}^3)$  with  $1 \leq p < \infty$  denotes the space of vector fields  $\boldsymbol{\xi} = (\xi_i) : \omega \rightarrow \mathbb{R}^3$  with components  $\xi_i$  in the usual Lebesgue space  $L^p(\omega)$ . It is equipped with the norm

$$\|\boldsymbol{\xi}\|_{L^p} := \left( \int_\omega |\boldsymbol{\xi}(x)|^p dx \right)^{1/p} \quad \text{for any } \boldsymbol{\xi} \in L^p(\omega; \mathbb{R}^3).$$

Likewise, the notation  $W^{1,p}(\omega; \mathbb{R}^3)$  denotes the space of vector fields  $\boldsymbol{\xi} = (\xi_i) : \omega \rightarrow \mathbb{R}^3$  with components  $\xi_i$  in the usual Sobolev space  $W^{1,p}(\omega)$ . It is

equipped with the norm

$$\|\boldsymbol{\xi}\|_{W^{1,p}} := \left( \|\boldsymbol{\xi}\|_{L^p}^p + \sum_{\alpha=1}^2 \|\partial_\alpha \boldsymbol{\xi}\|_{L^p}^p \right)^{1/p} \quad \text{for any } \boldsymbol{\xi} \in W^{1,p}(\omega; \mathbb{R}^3).$$

The space  $W^{1,\infty}(\omega; \mathbb{R}^3)$  denotes the space of vector fields  $\boldsymbol{\xi} = (\xi_i) : \omega \rightarrow \mathbb{R}^3$  with components  $\xi_i$  in the usual Sobolev space  $W^{1,\infty}(\omega)$ . The space  $W^{1,\infty}(\omega)$  consists of those Lipschitz continuous functions on  $\bar{\omega}$ .

Strong and weak convergences are respectively denoted  $\rightarrow$  and  $\rightharpoonup$ .

### 3 Definition of a $G^1$ shell

The purpose of this section is to define the regularity of the shell that we consider. To this end, we define a  $G^1$  shell which has been first introduced in (Anicic, 2001) and (Anicic, 2003). The term  $G^1$  stands for *First-Order Geometric Continuity*. This regularity allows us to take into account curvature discontinuities of  $S$  as well as the tangent plane continuity even if the tangent vectors are not continuous. Hence, if we consider a surface which is defined via smooth patches, we are only lead to match the unit normal vectors on their interfaces and not the tangent vectors. This makes for great versatility in practice. Besides, this regularity does not involve any Christoffel symbols.

The middle surface of the reference configuration of a shell is denoted by  $S := \boldsymbol{\varphi}(\omega)$  where

$$\boldsymbol{\varphi} \in W^{1,\infty}(\omega; \mathbb{R}^3). \quad (1)$$

The two vectors  $\mathbf{a}_\alpha := \partial_\alpha \boldsymbol{\varphi} \in L^\infty(\omega; \mathbb{R}^3)$  span the tangent plane to the surface  $S$ . We suppose that  $\boldsymbol{\varphi}$  satisfy the additional assumption

$$\text{ess inf}_\omega |\mathbf{a}_1 \wedge \mathbf{a}_2| > 0 \text{ and } \mathbf{a}_3 := \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|} \in W^{1,\infty}(\omega; \mathbb{R}^3), \quad (2)$$

where  $\mathbf{a}_3$  is the unit normal vector to the surface  $S$ .

The covariant components  $a_{\alpha\beta} \in L^\infty(\omega)$  of the first fundamental form and  $b_{\alpha\beta} \in L^\infty(\omega)$  of the second fundamental form of  $S$  are respectively defined by

$$a_{\alpha\beta} := \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \text{ and } b_{\alpha\beta} := -\mathbf{a}_\alpha \cdot \partial_\beta \mathbf{a}_3 = -\mathbf{a}_\beta \cdot \partial_\alpha \mathbf{a}_3.$$

The area element along  $S$  is  $\sqrt{a} dx$ , where

$$a := \det(a_{\alpha\beta}) = |\mathbf{a}_1 \wedge \mathbf{a}_2|^2 \in L^\infty(\omega).$$

Since

$$\operatorname{ess\,inf}_{\omega} a(x) > 0$$

the inverse of the matrix  $(a_{\alpha\beta})$ , which we denote by  $(a^{\alpha\beta})$ , belongs to  $L^\infty(\omega)$ . The contravariant basis  $\mathbf{a}^\alpha \in L^\infty(\omega; \mathbb{R}^3)$  is then defined by letting

$$\mathbf{a}^\alpha = a^{\alpha\beta} \mathbf{a}_\beta$$

and then satisfy

$$\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta_\beta^\alpha,$$

where  $\delta_\beta^\alpha$  is the Kronecker symbol.

The mixed components  $b_\alpha^\beta \in L^\infty(\omega)$  of the second fundamental form are defined by

$$b_\alpha^\beta := b_{\alpha\rho} a^{\rho\beta}.$$

The mean curvature  $H \in L^\infty(\omega)$  and the gaussian curvature  $K \in L^\infty(\omega)$  are respectively defined by

$$H := \frac{1}{2}(b_1^1 + b_2^2) = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \quad \text{and} \quad K := b_1^1 b_2^2 - b_1^2 b_2^1 = \frac{1}{R_1 R_2},$$

where the invariants  $1/R_\alpha$  are the principal curvatures of the middle surface  $S$  of the shell.

The reference configuration of a shell with thickness  $2\varepsilon > 0$  is the set

$$\{\Phi(x, z) := \boldsymbol{\varphi}(x) + z \mathbf{a}_3(x); \quad (x, z) \in \Omega := \omega \times (-\varepsilon, \varepsilon)\}.$$

The tangent vectors are respectively defined by

$$\mathbf{g}_\alpha := \partial_\alpha \Phi = \mathbf{a}_\alpha + z \partial_\alpha \mathbf{a}_3.$$

The two vectors  $\mathbf{g}_\alpha$  form a basis of the tangent plane of  $S$ ; its contravariant basis  $\mathbf{g}^\alpha$  is defined by

$$\mathbf{g}^\alpha \cdot \mathbf{g}_\beta = \delta_\beta^\alpha,$$

where  $\delta_\beta^\alpha$  is the Kronecker symbol. Hence

$$\begin{aligned} \det \nabla \Phi(x, z) &= \mathbf{g}_1(x, z) \wedge \mathbf{g}_2(x, z) \cdot \mathbf{a}_3(x) \\ &= (1 - 2H(x)z + K(x)z^2) \sqrt{a(x)} \\ &= \left(1 - \frac{z}{R_1(x)}\right) \left(1 - \frac{z}{R_2(x)}\right) \sqrt{a(x)}. \end{aligned}$$

In addition to the hypothesis (1)-(2) and in order that  $\mathbf{g}^\alpha \in L^\infty(\Omega; \mathbb{R}^3)$ , we also impose that  $\boldsymbol{\varphi}$  and  $\varepsilon$  satisfy the following assumption:

$$\operatorname{ess\,inf}_\Omega \det \nabla \Phi > 0. \quad (3)$$

To sum up, equivalently to the hypothesis (1)-(2)-(3), we will suppose that  $\boldsymbol{\varphi} \in G^1$  where

$$G^1 := \left\{ \boldsymbol{\varphi} \in W^{1,\infty}(\omega; \mathbb{R}^3); \operatorname{ess\,inf}_\omega |\mathbf{a}_1 \wedge \mathbf{a}_2| > 0, \right. \\ \left. \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|} \in W^{1,\infty}(\omega; \mathbb{R}^3), \max_{\alpha \in \{1,2\}} \left| \frac{\varepsilon}{R_\alpha} \right|_{\infty, \omega} < 1 \right\}.$$

## 4 An existence theorem

In this section, we define a notion of polyconvex and orientation-preserving stored energy function for a shell and we establish an existence theorem for the minimization problem of a nonlinearly elastic shell.

Let  $\omega$  be a domain in  $\mathbb{R}^2$ . For a given deformation  $\boldsymbol{\psi} \in W^{1,p}(\omega; \mathbb{R}^3)$ ,  $p \geq 2$ , we denote

$$a(\boldsymbol{\psi}) := \det(a_{\alpha\beta}(\boldsymbol{\psi})) = |\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}|^2$$

where  $a_{\alpha\beta}(\boldsymbol{\psi}) := \partial_\alpha \boldsymbol{\psi} \cdot \partial_\beta \boldsymbol{\psi}$  and if  $\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi} \neq 0$  a.e. in  $\omega$ , we denote

$$\mathbf{a}_3(\boldsymbol{\psi}) := \frac{\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}}{|\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}|}$$

the unit normal vector to the deformed surface  $\boldsymbol{\psi}(\omega)$ .

If  $\mathbf{a}_3(\boldsymbol{\psi}) \in W^{1,p}(\omega; \mathbb{R}^3)$ , we denote

$$H(\boldsymbol{\psi}) := \frac{1}{2} \left( \frac{1}{R_1(\boldsymbol{\psi})} + \frac{1}{R_2(\boldsymbol{\psi})} \right) \quad \text{and} \quad K(\boldsymbol{\psi}) := \frac{1}{R_1(\boldsymbol{\psi})R_2(\boldsymbol{\psi})}$$

the mean and gaussian curvature of the deformed surface  $\boldsymbol{\psi}(\omega)$  where  $1/R_1(\boldsymbol{\psi})$  and  $1/R_2(\boldsymbol{\psi})$  are the principal curvatures, namely the two eigenvalues of the matrix  $(b_\alpha^\beta(\boldsymbol{\psi}))$  defined as

$$b_\alpha^\beta(\boldsymbol{\psi}) := b_{\alpha\rho}(\boldsymbol{\psi})a^{\rho\beta}(\boldsymbol{\psi}), \quad b_{\alpha\rho}(\boldsymbol{\psi}) := -\partial_\alpha \mathbf{a}_3(\boldsymbol{\psi}) \cdot \partial_\rho \boldsymbol{\psi},$$

with the matrix  $(a^{\alpha\beta}(\boldsymbol{\psi}))$  the inverse of the matrix  $(a_{\alpha\beta}(\boldsymbol{\psi}))$ .

**Theorem 1.** Let  $\omega$  be a domain in  $\mathbb{R}^2$  and let  $\gamma_0$  be a non-empty relatively open subset of  $\gamma := \partial\omega$ . For  $p \geq 2$  and  $q > 1$ , we define the functional  $I : \mathbf{V}^\varepsilon \rightarrow \mathbb{R} \cup \{+\infty\}$  by letting

$$\begin{aligned} \mathbf{V}^\varepsilon := & \{ \boldsymbol{\psi} \in W^{1,p}(\omega; \mathbb{R}^3); \quad \sqrt{a(\boldsymbol{\psi})} \in L^q(\omega), \quad a(\boldsymbol{\psi}) \neq 0 \text{ a.e. in } \omega, \\ & \mathbf{a}_3(\boldsymbol{\psi}) \in W^{1,p}(\omega; \mathbb{R}^3), \quad \max \{ |\varepsilon/R_1(\boldsymbol{\psi})|, |\varepsilon/R_2(\boldsymbol{\psi})| \} < 1 \text{ a.e. in } \omega, \\ & \boldsymbol{\psi} = \boldsymbol{\varphi} \quad \text{and} \quad \mathbf{a}_3(\boldsymbol{\psi}) = \mathbf{a}_3 \quad d\gamma\text{-a.e. in } \gamma_0 \} \end{aligned}$$

and for each  $\boldsymbol{\psi} \in \mathbf{V}^\varepsilon$ ,

$$I(\boldsymbol{\psi}) := \int_{\omega} W(x, \boldsymbol{\psi}) \, dx - L(\boldsymbol{\psi}, \mathbf{a}_3(\boldsymbol{\psi})),$$

where  $L$  is a continuous linear form over the space  $W^{1,p}(\omega; \mathbb{R}^3) \times W^{1,p}(\omega; \mathbb{R}^3)$  and  $W : \omega \times \mathbf{V}^\varepsilon \rightarrow \mathbb{R}$  is a function with the following properties:

(a) *Polyconvexity:* For almost all  $x \in \omega$ , there exist a convex function  $\mathbb{W}(x, \cdot) : \mathbf{M} \rightarrow \mathbb{R}$  where

$$\mathbf{M} := \{ (\mathbf{A}, \mathbf{B}, a, b, c) \in (\mathbb{M}_{3 \times 2})^2 \times \mathbb{R}^3; \, a - |b| > 0 \text{ and } a - 2|b| + c > 0 \}$$

such that for almost all  $x \in \omega$

$$W(x, \boldsymbol{\psi}) = \mathbb{W}\left(x, \nabla \boldsymbol{\psi}(x), \nabla \mathbf{a}_3(\boldsymbol{\psi})(x), (1, \varepsilon H(\boldsymbol{\psi}(x)), \varepsilon^2 K(\boldsymbol{\psi}(x))) \sqrt{a(\boldsymbol{\psi}(x))}\right).$$

(b) *Measurability:* The function  $\mathbb{W}(\cdot, \mathbf{A}, \mathbf{B}, a, b, c) : \omega \rightarrow \mathbb{R}$  is measurable for all  $(\mathbf{A}, \mathbf{B}, a, b, c) \in \mathbf{M}$ .

(c) *Coerciveness:* There exist constants  $C_1 > 0$  and  $C_2$  such that

$$W(x, \boldsymbol{\psi}) \geq C_1 \{ |\nabla \boldsymbol{\psi}|^p + |\nabla \mathbf{a}_3(\boldsymbol{\psi})|^p + a(\boldsymbol{\psi})^{q/2} \} + C_2$$

for all  $\boldsymbol{\psi} \in \mathbf{V}^\varepsilon$  and almost all  $x \in \omega$ .

(d) *Orientation-preserving condition:*

$$\begin{aligned} W(x, \boldsymbol{\psi}) & \rightarrow \infty \text{ as } \{1 - 2\varepsilon H(\boldsymbol{\psi}(x)) + \varepsilon^2 K(\boldsymbol{\psi}(x))\} \sqrt{a(\boldsymbol{\psi}(x))} \rightarrow 0^+ \\ \text{and } W(x, \boldsymbol{\psi}) & \rightarrow \infty \text{ as } \{1 + 2\varepsilon H(\boldsymbol{\psi}(x)) + \varepsilon^2 K(\boldsymbol{\psi}(x))\} \sqrt{a(\boldsymbol{\psi}(x))} \rightarrow 0^+ \end{aligned}$$

for all  $\boldsymbol{\psi} \in \mathbf{V}^\varepsilon$  and almost all  $x \in \omega$ .

Assume that  $\inf_{\boldsymbol{\psi} \in \mathbf{V}^\varepsilon} I(\boldsymbol{\psi}) < \infty$ , then there exists at least one function  $\boldsymbol{\eta} \in \mathbf{V}^\varepsilon$  such that

$$I(\boldsymbol{\eta}) = \inf_{\boldsymbol{\psi} \in \mathbf{V}^\varepsilon} I(\boldsymbol{\psi}).$$

*Proof.* (i) The integrals  $\int_{\omega} W(x, \boldsymbol{\psi}) \, dx$  are well defined for all  $\boldsymbol{\psi} \in \mathbf{V}^{\varepsilon}$ . First, we note that the set  $\mathbf{M}$  is a convex open subset of  $(\mathbb{M}_{3 \times 2})^2 \times \mathbb{R}^3$ . Furthermore, each  $\boldsymbol{\psi} \in \mathbf{V}^{\varepsilon}$  satisfy  $a(\boldsymbol{\psi}(x)) > 0$  and  $|\varepsilon/R_{\alpha}(\boldsymbol{\psi}(x))| < 1$  for almost all  $x \in \omega$ , hence

$$\begin{aligned} |\varepsilon H(\boldsymbol{\psi}(x))| &= \frac{1}{2} \left| \frac{\varepsilon}{R_1(\boldsymbol{\psi}(x))} + \frac{\varepsilon}{R_2(\boldsymbol{\psi}(x))} \right| < 1, \\ 1 - 2\varepsilon H(\boldsymbol{\psi}(x)) + \varepsilon^2 K(\boldsymbol{\psi}(x)) &= \left(1 - \frac{\varepsilon}{R_1(\boldsymbol{\psi}(x))}\right) \left(1 - \frac{\varepsilon}{R_2(\boldsymbol{\psi}(x))}\right) > 0, \\ 1 + 2\varepsilon H(\boldsymbol{\psi}(x)) + \varepsilon^2 K(\boldsymbol{\psi}(x)) &= \left(1 + \frac{\varepsilon}{R_1(\boldsymbol{\psi}(x))}\right) \left(1 + \frac{\varepsilon}{R_2(\boldsymbol{\psi}(x))}\right) > 0. \end{aligned}$$

Thus,

$$\begin{aligned} |\varepsilon H(\boldsymbol{\psi}(x))| \sqrt{a(\boldsymbol{\psi}(x))} &< \sqrt{a(\boldsymbol{\psi}(x))} \\ \text{and } 2|\varepsilon H(\boldsymbol{\psi}(x))| \sqrt{a(\boldsymbol{\psi}(x))} &< \left(1 + \varepsilon^2 K(\boldsymbol{\psi}(x))\right) \sqrt{a(\boldsymbol{\psi}(x))} \end{aligned}$$

almost everywhere in  $\omega$ . It follows that almost everywhere in  $\omega$ ,

$$(\nabla \boldsymbol{\psi}(x), \nabla \mathbf{a}_3(\boldsymbol{\psi})(x), (1, \varepsilon H(\boldsymbol{\psi}(x)), \varepsilon^2 K(\boldsymbol{\psi}(x))) \sqrt{a(\boldsymbol{\psi}(x))}) \in \mathbf{M}.$$

Furthermore, for almost all  $x \in \omega$ , the function  $\mathbb{W}(x, \cdot) : \mathbf{M} \rightarrow \mathbb{R}$  is continuous (as a convex and real-valued function on a convex open subset of a finite-dimensional space (Ciarlet, 2013, Theorem 2.17-1)). For all  $(\mathbf{A}, \mathbf{B}, a, b, c) \in \mathbf{M}$ , the function  $\mathbb{W}(\cdot, \mathbf{A}, \mathbf{B}, a, b, c) : \omega \rightarrow \mathbb{R}$  is measurable, and  $\mathbf{M}$  is a Borel set. Therefore the function  $\mathbb{W} : \omega \times \mathbf{M} \rightarrow \mathbb{R}$  is a Carathéodory function, and consequently the function

$$x \in \omega \rightarrow \mathbb{W}(x, \nabla \boldsymbol{\psi}(x), \nabla \mathbf{a}_3(\boldsymbol{\psi})(x), \alpha(x), \beta(x), \gamma(x)) \in \mathbb{R}$$

with  $\alpha(x) := \sqrt{a(\boldsymbol{\psi}(x))}$ ,  $\beta(x) := \varepsilon H(\boldsymbol{\psi}(x))\alpha(x)$  and  $\gamma(x) := \varepsilon^2 K(\boldsymbol{\psi}(x))\alpha(x)$  is measurable for each  $\boldsymbol{\psi} \in \mathbf{V}^{\varepsilon}$ . The function  $W$  being in addition bounded from below (by the coerciveness inequality), the integral

$$\int_{\omega} W(x, \boldsymbol{\psi}) \, dx = \int_{\omega} \mathbb{W}(x, \nabla \boldsymbol{\psi}(x), \nabla \mathbf{a}_3(\boldsymbol{\psi})(x), \alpha(x), \beta(x), \gamma(x)) \, dx$$

is thus a well-defined extended real number in the interval  $[C_2 \text{ area } \omega, \infty]$  for each  $\boldsymbol{\psi} \in \mathbf{V}^{\varepsilon}$ .



(ii) We find a lower bound for  $I(\psi)$  when  $\psi \in \mathbf{V}^\varepsilon$ .

From the assumed coerciveness (c) of the function  $W$  and the assumed continuity of the linear form  $L$ , we infer that there exists a constant  $C_3 > 0$  such that

$$I(\psi) \geq C_1 \int_{\omega} \{|\nabla \psi|^p + |\nabla \mathbf{a}_3(\psi)|^p + a(\psi)^{q/2}\} dx + C_2 \text{ area } \omega \\ - C_3(\|\psi\|_{1,p,\omega} + \|\mathbf{a}_3(\psi)\|_{1,p,\omega}) \text{ for all } \psi \in \mathbf{V}^\varepsilon.$$

Combining the boundary conditions  $\psi = \varphi$  and  $\mathbf{a}_3(\psi) = \mathbf{a}_3$  on  $\gamma_0$  with the generalized Poincaré inequality, we thus conclude that there exist constants  $C_4 > 0$  and  $C_5$  such that

$$I(\psi) \geq C_4\{\|\psi\|_{1,p,\omega}^p + \|\mathbf{a}_3(\psi)\|_{1,p,\omega}^p + \|\sqrt{a(\psi)}\|_{0,q,\omega}^q\} + C_5 \text{ for all } \psi \in \mathbf{V}^\varepsilon.$$

(iii) We show that if  $(\eta^k)$  is a sequence with  $\eta^k \in \mathbf{V}^\varepsilon$  for all  $k$  for which there exist  $\eta \in W^{1,p}(\omega; \mathbb{R}^3)$ ,  $\kappa \in W^{1,p}(\omega; \mathbb{R}^3)$ ,  $(\xi_1, \xi_2, \xi_3) \in (L^q(\omega; \mathbb{R}^3))^3$  and  $(\alpha_1, \alpha_2, \alpha_3) \in (L^q(\omega))^3$  such that

$$\begin{aligned} \eta^k &\rightharpoonup \eta \text{ in } W^{1,p}(\omega; \mathbb{R}^3), \quad \mathbf{a}_3(\eta^k) \rightharpoonup \kappa \text{ in } W^{1,p}(\omega; \mathbb{R}^3) \\ \partial_1 \eta^k \wedge \partial_2 \eta^k &\rightharpoonup \xi_1 \text{ in } L^q(\omega; \mathbb{R}^3), \quad \sqrt{a(\eta^k)} \rightharpoonup \alpha_1 \text{ in } L^q(\omega) \\ H(\eta^k) \partial_1 \eta^k \wedge \partial_2 \eta^k &\rightharpoonup \xi_2 \text{ in } L^q(\omega; \mathbb{R}^3), \quad H(\eta^k) \sqrt{a(\eta^k)} \rightharpoonup \alpha_2 \text{ in } L^q(\omega) \\ K(\eta^k) \partial_1 \eta^k \wedge \partial_2 \eta^k &\rightharpoonup \xi_3 \text{ in } L^q(\omega; \mathbb{R}^3), \quad K(\eta^k) \sqrt{a(\eta^k)} \rightharpoonup \alpha_3 \text{ in } L^q(\omega) \end{aligned}$$

then almost everywhere in  $\omega$

$$\begin{aligned} \kappa &= \mathbf{a}_3(\eta), \quad \max\{|\varepsilon/R_1(\eta)|, |\varepsilon/R_2(\eta)|\} \leq 1, \\ \xi_1 &= \partial_1 \eta \wedge \partial_2 \eta, \quad \xi_2 = H(\eta) \partial_1 \eta \wedge \partial_2 \eta, \quad \xi_3 = K(\eta) \partial_1 \eta \wedge \partial_2 \eta \\ \alpha_1 &= \sqrt{a(\eta)}, \quad \alpha_2 = H(\eta) \sqrt{a(\eta)} \quad \text{and} \quad \alpha_3 = K(\eta) \sqrt{a(\eta)}. \end{aligned}$$

To prove this assertion, we begin by showing that  $\kappa = \mathbf{a}_3(\eta)$ . By the Rellich-Kondrašov compact imbedding theorem  $W^{1,p}(\omega; \mathbb{R}^3) \Subset L^r(\omega; \mathbb{R}^3)$  for all  $r$  with  $1 \leq r < \infty$  (Adams and Fournier, 2003, p. 168),

$$\begin{aligned} \mathbf{a}_3(\eta^k) &\rightarrow \kappa \text{ in } L^{p'}(\omega; \mathbb{R}^3), \quad \frac{1}{p} + \frac{1}{p'} = 1 \\ \text{and } \mathbf{a}_3(\eta^k) &\rightarrow \kappa \text{ in } L^2(\omega; \mathbb{R}^3). \end{aligned}$$

Hence

$$\begin{aligned} \partial_\alpha \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) &\rightharpoonup \partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{\kappa} \text{ in } L^1(\omega) \\ \text{and } |\mathbf{a}_3(\boldsymbol{\eta}^k)|^2 &\rightarrow |\boldsymbol{\kappa}|^2 \text{ in } L^1(\omega). \end{aligned}$$

Since for all  $k$ ,  $\partial_\alpha \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) = 0$  and  $|\mathbf{a}_3(\boldsymbol{\eta}^k)| = 1$ , it follows that

$$\partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{\kappa} = 0 \text{ and } |\boldsymbol{\kappa}| = 1.$$

In order to prove that  $\boldsymbol{\kappa} = \mathbf{a}_3(\boldsymbol{\eta})$ , it remains to show that

$$\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \boldsymbol{\kappa} \geq 0 \text{ a.e. on } \omega.$$

In order to simplify the notations, we define for all  $\boldsymbol{\varphi}_1 \in W^{1,p}(\omega; \mathbb{R}^3)$  and all  $\boldsymbol{\varphi}_2 \in W^{1,p}(\omega; \mathbb{R}^3)$

$$[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] := \frac{1}{2} (\partial_1 \boldsymbol{\varphi}_1 \wedge \partial_2 \boldsymbol{\varphi}_2 + \partial_1 \boldsymbol{\varphi}_2 \wedge \partial_2 \boldsymbol{\varphi}_1) \in L^{p/2}(\omega; \mathbb{R}^3).$$

We now notice that  $[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]$  can be rewritten in the following form

$$[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] = \frac{1}{4} \{ \partial_1 (\boldsymbol{\varphi}_1 \wedge \partial_2 \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_2 \wedge \partial_2 \boldsymbol{\varphi}_1) + \partial_2 (\partial_1 \boldsymbol{\varphi}_1 \wedge \boldsymbol{\varphi}_2 + \partial_1 \boldsymbol{\varphi}_2 \wedge \boldsymbol{\varphi}_1) \}.$$

Hence, if  $(\boldsymbol{\varphi}_1^k, \boldsymbol{\varphi}_2^k)$  is a sequence of  $W^{1,p}(\omega; \mathbb{R}^3)$ ,  $p \geq 2$ , which converges weakly to  $(\boldsymbol{l}_1, \boldsymbol{l}_2) \in W^{1,p}(\omega; \mathbb{R}^3)$ , then by the Rellich-Kondrašov compact imbedding theorem  $W^{1,p}(\omega; \mathbb{R}^3) \Subset L^r(\omega; \mathbb{R}^3)$  for all  $r$  with  $1 \leq r < \infty$ , it follows that for all  $\alpha \in \{1, 2\}$

$$\boldsymbol{\varphi}_\alpha^k \rightarrow \boldsymbol{l}_\alpha \text{ in } L^{p'}(\omega; \mathbb{R}^3), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Therefore,

$$[\boldsymbol{\varphi}_1^k, \boldsymbol{\varphi}_2^k] \rightharpoonup [\boldsymbol{l}_1, \boldsymbol{l}_2] \text{ in } \mathcal{D}'(\omega; \mathbb{R}^3).$$

Now by applying this result to the sequence  $(\boldsymbol{\eta}^k)$  which converges weakly to  $\boldsymbol{\eta}$  in  $W^{1,p}(\omega; \mathbb{R}^3)$ , it follows that

$$[\boldsymbol{\eta}^k, \boldsymbol{\eta}^k] := \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \rightharpoonup [\boldsymbol{\eta}, \boldsymbol{\eta}] := \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \text{ in } \mathcal{D}'(\omega; \mathbb{R}^3)$$

Hence

$$\boldsymbol{\xi}_1 = \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \quad \text{and} \quad \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \rightharpoonup \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \text{ in } L^q(\omega; \mathbb{R}^3).$$

Since

$$\mathbf{a}_3(\boldsymbol{\eta}^k) \rightarrow \boldsymbol{\kappa} \text{ in } L^{q'}(\omega; \mathbb{R}^3), \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

it follows that

$$\sqrt{a(\boldsymbol{\eta}^k)} = \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) \rightharpoonup \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \boldsymbol{\kappa} \text{ in } L^1(\omega).$$

Since for all  $k$ ,  $\partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) > 0$  (the inequality  $\geq 0$  would suffice here) then

$$\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \boldsymbol{\kappa} \geq 0 \text{ a.e. in } \omega.$$

Although this implication is standard, we provide a proof for completeness. Assume that  $\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \boldsymbol{\kappa} < 0$  on a subset  $A$  of  $\omega$  with  $\text{dx-meas} A > 0$ . Then

$$\int_A \sqrt{a(\boldsymbol{\eta}^k)} \, dx = \int_A \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) \, dx \rightarrow \int_A \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \boldsymbol{\kappa} \, dx \geq 0$$

by definition of weak convergence (the characteristic function of the set  $A$  belongs to  $L^\infty(\omega)$ ). But this contradicts the inequality

$$\int_A \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \boldsymbol{\kappa} < 0.$$

Combining the three relations,

$$\partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{\kappa} = 0, \quad |\boldsymbol{\kappa}| = 1, \quad \text{and} \quad \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \boldsymbol{\kappa} \geq 0 \text{ a.e. in } \omega,$$

we infer that

$$\boldsymbol{\kappa} = \frac{\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta}}{|\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta}|} = \mathbf{a}_3(\boldsymbol{\eta}) \quad \text{and} \quad \alpha_1 = \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \boldsymbol{\kappa} = \sqrt{a(\boldsymbol{\eta})}.$$

To sum up, we have proved

$$\begin{aligned} \mathbf{a}_3(\boldsymbol{\eta}^k) &\rightharpoonup \mathbf{a}_3(\boldsymbol{\eta}) \text{ in } W^{1,p}(\omega; \mathbb{R}^3) \\ \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k &\rightharpoonup \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \text{ in } L^q(\omega; \mathbb{R}^3) \\ \sqrt{a(\boldsymbol{\eta}^k)} &\rightharpoonup \sqrt{a(\boldsymbol{\eta})} \text{ in } L^q(\omega). \end{aligned}$$

Since

$$(\boldsymbol{\eta}^k, \mathbf{a}_3(\boldsymbol{\eta}^k)) \rightharpoonup (\boldsymbol{\eta}, \mathbf{a}_3(\boldsymbol{\eta})) \text{ in } W^{1,p}(\omega; \mathbb{R}^3)$$

it follows that

$$\begin{aligned} [\boldsymbol{\eta}^k, \mathbf{a}_3(\boldsymbol{\eta}^k)] &:= -H(\boldsymbol{\eta}^k) \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \rightharpoonup [\boldsymbol{\eta}, \mathbf{a}_3(\boldsymbol{\eta})] := -H(\boldsymbol{\eta}) \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \\ [\mathbf{a}_3(\boldsymbol{\eta}^k), \mathbf{a}_3(\boldsymbol{\eta}^k)] &:= K(\boldsymbol{\eta}^k) \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \rightharpoonup [\mathbf{a}_3(\boldsymbol{\eta}), \mathbf{a}_3(\boldsymbol{\eta})] := K(\boldsymbol{\eta}) \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \end{aligned}$$

in  $\mathcal{D}'(\omega; \mathbb{R}^3)$ . Hence  $\boldsymbol{\xi}_2 = H(\boldsymbol{\eta}) \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta}$ ,  $\boldsymbol{\xi}_3 = K(\boldsymbol{\eta}) \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta}$  and

$$\begin{aligned} H(\boldsymbol{\eta}^k) \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k &\rightharpoonup H(\boldsymbol{\eta}) \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \text{ in } L^q(\omega; \mathbb{R}^3) \\ K(\boldsymbol{\eta}^k) \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k &\rightharpoonup K(\boldsymbol{\eta}) \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \text{ in } L^q(\omega; \mathbb{R}^3). \end{aligned}$$

Since

$$\mathbf{a}_3(\boldsymbol{\eta}^k) \rightarrow \mathbf{a}_3(\boldsymbol{\eta}) \text{ in } L^{q'}(\omega; \mathbb{R}^3), \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

it follows that

$$\begin{aligned} H(\boldsymbol{\eta}^k) \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) &\rightharpoonup H(\boldsymbol{\eta}) \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \mathbf{a}_3(\boldsymbol{\eta}) = H(\boldsymbol{\eta}) \sqrt{a(\boldsymbol{\eta})} \\ K(\boldsymbol{\eta}^k) \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) &\rightharpoonup K(\boldsymbol{\eta}) \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \cdot \mathbf{a}_3(\boldsymbol{\eta}) = K(\boldsymbol{\eta}) \sqrt{a(\boldsymbol{\eta})} \end{aligned}$$

in  $L^1(\omega)$ . But for all  $k$

$$\begin{aligned} H(\boldsymbol{\eta}^k) \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) &= H(\boldsymbol{\eta}^k) \sqrt{a(\boldsymbol{\eta}^k)}, \\ \text{and } K(\boldsymbol{\eta}^k) \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) &= K(\boldsymbol{\eta}^k) \sqrt{a(\boldsymbol{\eta}^k)}, \end{aligned}$$

then  $\alpha_2 = H(\boldsymbol{\eta}) \sqrt{a(\boldsymbol{\eta})}$ ,  $\alpha_3 = K(\boldsymbol{\eta}) \sqrt{a(\boldsymbol{\eta})}$  and

$$\begin{aligned} H(\boldsymbol{\eta}^k) \sqrt{a(\boldsymbol{\eta}^k)} &\rightharpoonup H(\boldsymbol{\eta}) \sqrt{a(\boldsymbol{\eta})} \text{ in } L^q(\omega) \\ \text{and } K(\boldsymbol{\eta}^k) \sqrt{a(\boldsymbol{\eta}^k)} &\rightharpoonup K(\boldsymbol{\eta}) \sqrt{a(\boldsymbol{\eta})} \text{ in } L^q(\omega). \end{aligned}$$

It remains to show that for all  $\alpha \in \{1, 2\}$ ,

$$\left| \frac{\varepsilon}{R_\alpha(\boldsymbol{\eta})} \right| \leq 1 \text{ a.e. in } \omega.$$

Combining all the previous relations lead to the following weak convergence in  $L^q(\omega)$  for all  $d \in \{-1, 1\}$

$$\begin{aligned} (1 - d\varepsilon H(\boldsymbol{\eta}^k)) \sqrt{a(\boldsymbol{\eta}^k)} &\rightharpoonup (1 - d\varepsilon H(\boldsymbol{\eta})) \sqrt{a(\boldsymbol{\eta})}, \\ (1 - 2d\varepsilon H(\boldsymbol{\eta}^k) + \varepsilon^2 K(\boldsymbol{\eta}^k)) \sqrt{a(\boldsymbol{\eta}^k)} &\rightharpoonup (1 - 2d\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta})) \sqrt{a(\boldsymbol{\eta})}. \end{aligned}$$

Since for all  $k$  and all  $\alpha \in \{1, 2\}$ ,

$$\sqrt{a(\boldsymbol{\eta}^k)} > 0 \text{ a.e. in } \omega \text{ and } \left| \frac{\varepsilon}{R_\alpha(\boldsymbol{\eta}^k)} \right| < 1 \text{ a.e. in } \omega,$$

then for all  $k$  and all  $d \in \{-1, 1\}$ ,

$$(1 - d\varepsilon H(\boldsymbol{\eta}^k))\sqrt{a(\boldsymbol{\eta}^k)} > 0 \text{ a.e. in } \omega$$

$$\text{and } (1 - 2d\varepsilon H(\boldsymbol{\eta}^k) + \varepsilon^2 K(\boldsymbol{\eta}^k))\sqrt{a(\boldsymbol{\eta}^k)} > 0 \text{ a.e. in } \omega.$$

By passing to the weak limit in  $L^q(\omega)$ , it follows that for all  $d \in \{-1, 1\}$ ,

$$(1 - d\varepsilon H(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})} \geq 0 \text{ a.e. in } \omega$$

$$\text{and } (1 - 2d\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})} \geq 0 \text{ a.e. in } \omega.$$

Hence for all  $\alpha \in \{1, 2\}$

$$\left| \frac{\varepsilon}{R_\alpha(\boldsymbol{\eta})} \right| \leq 1 \text{ a.e. in } \omega.$$

(iv) Let  $(\boldsymbol{\eta}^k)$  be an infimizing sequence for the functional  $I$ , i.e., a sequence that satisfies

$$\boldsymbol{\eta}^k \in \mathbf{V}^\varepsilon \text{ for all } k, \quad \text{and} \quad \lim_{k \rightarrow \infty} I(\boldsymbol{\eta}^k) = \inf_{\boldsymbol{\psi} \in \mathbf{V}^\varepsilon} I(\boldsymbol{\psi}).$$

By assumption,  $\inf_{\boldsymbol{\psi} \in \mathbf{V}^\varepsilon} I(\boldsymbol{\psi}) < \infty$ , and thus, by the coerciveness property (c), the sequence  $(\boldsymbol{\eta}^k, \mathbf{a}_3(\boldsymbol{\eta}^k))$  is bounded in  $(W^{1,p}(\omega; \mathbb{R}^3))^2$  and the sequence  $\sqrt{a(\boldsymbol{\eta}^k)}$  is bounded in  $L^q(\omega)$ . Since

$$\sqrt{a(\boldsymbol{\eta}^k)} = \partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k \cdot \mathbf{a}_3(\boldsymbol{\eta}^k) = |\partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k|$$

we infer that the sequence  $(\partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k)$  is bounded in  $L^q(\omega; \mathbb{R}^3)$ . As the sequences  $(1/R_1(\boldsymbol{\eta}^k))$  and  $(1/R_2(\boldsymbol{\eta}^k))$  are bounded in  $L^\infty(\omega)$ , it follows that the sequences  $(H(\boldsymbol{\eta}^k)\partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k)$  and  $(K(\boldsymbol{\eta}^k)\partial_1 \boldsymbol{\eta}^k \wedge \partial_2 \boldsymbol{\eta}^k)$  are bounded in  $L^q(\omega; \mathbb{R}^3)$  on the one hand and on the other hand that the sequences  $(H(\boldsymbol{\eta}^k)\sqrt{a(\boldsymbol{\eta}^k)})$  and  $(K(\boldsymbol{\eta}^k)\sqrt{a(\boldsymbol{\eta}^k)})$  are bounded in  $L^q(\omega)$ .

Hence, there exists a subsequence  $(\boldsymbol{\eta}^\ell, \mathbf{a}_3(\boldsymbol{\eta}^\ell))$  that converges weakly to an element  $(\boldsymbol{\eta}, \boldsymbol{\kappa})$  in  $W^{1,p}(\omega; \mathbb{R}^3)$ . There exists also six other subsequences

$$(\partial_1 \boldsymbol{\eta}^\ell \wedge \partial_2 \boldsymbol{\eta}^\ell), \quad (H(\boldsymbol{\eta}^\ell)\partial_1 \boldsymbol{\eta}^\ell \wedge \partial_2 \boldsymbol{\eta}^\ell), \quad (K(\boldsymbol{\eta}^\ell)\partial_1 \boldsymbol{\eta}^\ell \wedge \partial_2 \boldsymbol{\eta}^\ell)$$

which converge weakly to  $\xi_1, \xi_2, \xi_3$  in  $L^q(\omega; \mathbb{R}^3)$  respectively and

$$(\sqrt{a(\boldsymbol{\eta}^\ell)}), \quad (H(\boldsymbol{\eta}^\ell)\sqrt{a(\boldsymbol{\eta}^\ell)}), \quad (K(\boldsymbol{\eta}^\ell)\sqrt{a(\boldsymbol{\eta}^\ell)})$$

which converge weakly to  $\alpha_1, \alpha_2, \alpha_3$  in  $L^q(\omega)$  respectively. Then by (iii), we infer that for all  $\alpha \in \{1, 2\}$ ,

$$\left| \frac{\varepsilon}{R_\alpha(\boldsymbol{\eta})} \right| \leq 1 \text{ a.e. in } \omega$$

and

$$\begin{aligned} \boldsymbol{\eta}^\ell &\rightharpoonup \boldsymbol{\eta} \text{ in } W^{1,p}(\omega; \mathbb{R}^3) \\ \mathbf{a}_3(\boldsymbol{\eta}^\ell) &\rightharpoonup \mathbf{a}_3(\boldsymbol{\eta}) \text{ in } W^{1,p}(\omega; \mathbb{R}^3) \\ \partial_1 \boldsymbol{\eta}^\ell \wedge \partial_2 \boldsymbol{\eta}^\ell &\rightharpoonup \partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \text{ in } L^q(\omega; \mathbb{R}^3) \\ \sqrt{a(\boldsymbol{\eta}^\ell)} &\rightharpoonup \sqrt{a(\boldsymbol{\eta})} \text{ in } L^q(\omega) \\ H(\boldsymbol{\eta}^\ell)\sqrt{a(\boldsymbol{\eta}^\ell)} &\rightharpoonup H(\boldsymbol{\eta})\sqrt{a(\boldsymbol{\eta})} \text{ in } L^q(\omega) \\ K(\boldsymbol{\eta}^\ell)\sqrt{a(\boldsymbol{\eta}^\ell)} &\rightharpoonup K(\boldsymbol{\eta})\sqrt{a(\boldsymbol{\eta})} \text{ in } L^q(\omega). \end{aligned}$$

In order to prove that  $\boldsymbol{\eta} \in \mathbf{V}^\varepsilon$ , it thus remains to establish that

$$\boldsymbol{\eta}|_{\gamma_0} = \boldsymbol{\varphi}, \quad \mathbf{a}_3(\boldsymbol{\eta})|_{\gamma_0} = \mathbf{a}_3,$$

$$\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \neq 0 \text{ a.e. in } \omega \text{ and for all } \alpha \in \{1, 2\}, \left| \frac{\varepsilon}{R_\alpha(\boldsymbol{\eta})} \right| \neq 1 \text{ a.e. in } \omega.$$

Since the trace operator from  $W^{1,p}(\omega)$  into  $L^p(\gamma_0)$  is continuous with respect to the strong topologies of both spaces, it remains so with respect to the weak topologies of both spaces. Hence, we infer from the weak convergence

$$\boldsymbol{\eta}^\ell \rightharpoonup \boldsymbol{\eta} \text{ and } \mathbf{a}_3(\boldsymbol{\eta}^\ell) \rightharpoonup \mathbf{a}_3(\boldsymbol{\eta}) \text{ in } W^{1,p}(\omega; \mathbb{R}^3)$$

that

$$\boldsymbol{\eta}^\ell|_{\gamma_0} \rightarrow \boldsymbol{\eta}|_{\gamma_0} \text{ and } \mathbf{a}_3(\boldsymbol{\eta}^\ell)|_{\gamma_0} \rightarrow \mathbf{a}_3(\boldsymbol{\eta})|_{\gamma_0} \text{ in } L^p(\gamma_0; \mathbb{R}^3)$$

and thus

$$\boldsymbol{\eta}|_{\gamma_0} = \boldsymbol{\varphi} \quad \text{and} \quad \mathbf{a}_3(\boldsymbol{\eta})|_{\gamma_0} = \mathbf{a}_3$$

since  $\boldsymbol{\eta}^\ell|_{\gamma_0} = \boldsymbol{\varphi}$  and  $\mathbf{a}_3(\boldsymbol{\eta}^\ell)|_{\gamma_0} = \mathbf{a}_3$  for all  $\ell$ .

In order to prove that

$$\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \neq 0 \text{ a.e. in } \omega \text{ and for all } \alpha \in \{1, 2\}, \left| \frac{\varepsilon}{R_\alpha(\boldsymbol{\eta})} \right| \neq 1 \text{ a.e. in } \omega,$$

it suffices to show that

$$(1 - 2\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})} \neq 0 \text{ a.e. in } \omega$$

$$\text{and } (1 + 2\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})} \neq 0 \text{ a.e. in } \omega.$$

Assume that  $(1 - 2\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})} = 0$  on a subset  $A$  of  $\omega$  with  $\text{dx-meas } A > 0$ . Since  $(1 - 2\varepsilon H(\boldsymbol{\eta}^\ell) + \varepsilon^2 K(\boldsymbol{\eta}^\ell))\sqrt{a(\boldsymbol{\eta}^\ell)} > 0$  a.e. on  $A$  and

$$(1 - 2\varepsilon H(\boldsymbol{\eta}^\ell) + \varepsilon^2 K(\boldsymbol{\eta}^\ell))\sqrt{a(\boldsymbol{\eta}^\ell)} \rightharpoonup (1 - 2\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})}$$

in  $L^q(\omega)$ , then

$$\int_A (1 - 2\varepsilon H(\boldsymbol{\eta}^\ell) + \varepsilon^2 K(\boldsymbol{\eta}^\ell))\sqrt{a(\boldsymbol{\eta}^\ell)} dx \rightarrow \int_A (1 - 2\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})} dx$$

bt the definition of weak convergence (the characteristic function of the set  $A$  belongs to the dual space of  $L^q(\omega)$ ), hence

$$(1 - 2\varepsilon H(\boldsymbol{\eta}^\ell) + \varepsilon^2 K(\boldsymbol{\eta}^\ell))\sqrt{a(\boldsymbol{\eta}^\ell)} \rightarrow 0 \text{ in } L^1(A).$$

Therefore there exists a subsequence  $(\boldsymbol{\eta}^m)$  of  $(\boldsymbol{\eta}^\ell)$  such that

$$(1 - 2\varepsilon H(\boldsymbol{\eta}^m(x)) + \varepsilon^2 K(\boldsymbol{\eta}^m(x)))\sqrt{a(\boldsymbol{\eta}^m(x))} \rightarrow 0 \text{ for almost all } x \in A.$$

Consider next the sequence of measurable functions  $(f^m)$  defined by

$$f^m: x \in A \rightarrow f^m(x) := W(x, \boldsymbol{\eta}^m).$$

Since  $f^m \geq C_2$  for all  $m$ , can apply Fatou's lemma, which shows that

$$\int_A \liminf_{m \rightarrow \infty} f^m(x) dx \leq \liminf_{m \rightarrow \infty} \int_A f^m(x) dx$$

on the one hand. On the other hand, the behavior of the function  $W$  as

$$(1 - 2\varepsilon H(\boldsymbol{\eta}(x)) + \varepsilon^2 K(\boldsymbol{\eta}(x)))\sqrt{a(\boldsymbol{\eta}(x))} \rightarrow 0^+$$

(assumption (d)) implies that

$$\liminf_{m \rightarrow \infty} f^m(x) = \lim_{m \rightarrow \infty} W(x, \boldsymbol{\eta}^m) = \infty \text{ for almost all } x \in A,$$

and thus

$$\lim_{m \rightarrow \infty} \int_A f^m(x) \, dx = \lim_{m \rightarrow \infty} \int_A W(x, \boldsymbol{\eta}^m) \, dx = \infty.$$

But this last relation contradicts the relation

$$\lim_{m \rightarrow \infty} I(\boldsymbol{\eta}^m) = \inf_{\boldsymbol{\psi} \in \mathbf{V}^\varepsilon} I(\boldsymbol{\psi}) < \infty$$

and the inequalities

$$\begin{aligned} I(\boldsymbol{\eta}^m) &\geq \int_A W(x, \boldsymbol{\eta}^m) \, dx + C_2 \text{ area } (\omega - A) \\ &\quad - C_3(\|\boldsymbol{\eta}^m\|_{1,p,\omega} + \|\mathbf{a}_3(\boldsymbol{\eta}^m)\|_{1,p,\omega}) \end{aligned}$$

(a weakly convergent sequence is bounded). Hence

$$(1 - 2\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})} \neq 0 \text{ a.e. in } \omega,$$

thus

$$\partial_1 \boldsymbol{\eta} \wedge \partial_2 \boldsymbol{\eta} \neq 0 \text{ a.e. in } \omega \text{ and for all } \alpha \in \{1, 2\}, \frac{\varepsilon}{R_\alpha(\boldsymbol{\eta})} \neq 1 \text{ a.e. in } \omega.$$

We proceed in the same way to prove

$$(1 + 2\varepsilon H(\boldsymbol{\eta}) + \varepsilon^2 K(\boldsymbol{\eta}))\sqrt{a(\boldsymbol{\eta})} \neq 0 \text{ a.e. in } \omega,$$

thus we infer in addition that for all  $\alpha \in \{1, 2\}$ ,

$$\frac{\varepsilon}{R_\alpha(\boldsymbol{\eta})} \neq -1 \text{ a.e. in } \omega.$$

To sum up, we have proved that  $\boldsymbol{\eta} \in \mathbf{V}^\varepsilon$ .

(v) *Finally, we show that*

$$\int_\omega W(x, \boldsymbol{\eta}) \, dx \leq \liminf_{\ell \rightarrow \infty} \int_\omega W(x, \boldsymbol{\eta}^\ell) \, dx.$$



By the definition of the limit inferior, we must show that, given any subsequence  $(\boldsymbol{\eta}^m)$  of  $(\boldsymbol{\eta}^\ell)$  such that the sequence  $(\int_\omega W(x, \boldsymbol{\eta}^m) dx)$  converges, then

$$\int_\omega W(x, \boldsymbol{\eta}) dx \leq \lim_{\ell \rightarrow \infty} \int_\omega W(x, \boldsymbol{\eta}^m) dx.$$

So, let us consider such a subsequence. Using the result of part (iv) and the Banach-Saks-Mazur theorem (Ciarlet, 2013, Theorem 5.13-1(c)), we infer that for each  $m$ , there exists integers  $j(m) \geq m$  and numbers  $\mu_t^m$ ,  $m \leq t \leq j(m)$ , such that

$$\begin{aligned} \mu_t^m &\geq 0, \quad \sum_{t=m}^{j(m)} \mu_t^m = 1, \\ \mathbf{D}^m &:= \sum_{t=m}^{j(m)} \mu_t^m (\nabla \boldsymbol{\eta}^t, \nabla \mathbf{a}_3(\boldsymbol{\eta}^t), (1, \varepsilon H(\boldsymbol{\eta}^t), \varepsilon^2 K(\boldsymbol{\eta}^t)) \sqrt{a(\boldsymbol{\eta}^t)}) \\ &\xrightarrow{m \rightarrow \infty} (\nabla \boldsymbol{\eta}, \nabla \mathbf{a}_3(\boldsymbol{\eta}), (1, \varepsilon H(\boldsymbol{\eta}), \varepsilon^2 K(\boldsymbol{\eta})) \sqrt{a(\boldsymbol{\eta})}) \end{aligned}$$

in  $(L^p(\omega; \mathbb{R}^3))^2 \times (L^q(\omega))^3$ . Hence there exists a subsequence  $(\mathbf{D}^n)$  of  $(\mathbf{D}^m)$  such that, for almost all  $x \in \omega$ ,

$$\begin{aligned} &\sum_{t=n}^{j(n)} \mu_t^n (\nabla \boldsymbol{\eta}^t(x), \nabla \mathbf{a}_3(\boldsymbol{\eta}^t(x)), (1, \varepsilon H(\boldsymbol{\eta}^t(x), \varepsilon^2 K(\boldsymbol{\eta}^t(x)) \sqrt{a(\boldsymbol{\eta}^t(x))})) \\ &\xrightarrow{n \rightarrow \infty} (\nabla \boldsymbol{\eta}(x), \nabla \mathbf{a}_3(\boldsymbol{\eta}(x)), (1, \varepsilon H(\boldsymbol{\eta}(x), \varepsilon^2 K(\boldsymbol{\eta}(x)) \sqrt{a(\boldsymbol{\eta}(x))})). \end{aligned}$$

Since the function  $\mathbb{W}(x, \cdot)$  is continuous on the set

$$\mathbf{M} := \{(\mathbf{A}, \mathbf{B}, a, b, c) \in (\mathbb{M}_{3 \times 2})^2 \times \mathbb{R}^3, a - |b| > 0 \text{ and } a - 2|b| + c > 0\}$$

for almost all  $x \in \omega$ , and since  $a(\boldsymbol{\eta}) > 0$  a.e. in  $\omega$  and  $|\varepsilon/R_\alpha(\boldsymbol{\eta})| < 1$  a.e. in  $\omega$ , it follows that for almost all  $x \in \omega$

$$(\nabla \boldsymbol{\eta}(x), \nabla \mathbf{a}_3(\boldsymbol{\eta}(x)), (1, \varepsilon H(\boldsymbol{\eta}(x), \varepsilon^2 K(\boldsymbol{\eta}(x)) \sqrt{a(\boldsymbol{\eta}(x))})) \in \mathbf{M}.$$

Then for almost all  $x \in \omega$ ,

$$\begin{aligned} W(x, \boldsymbol{\eta}) &= \mathbb{W}(x, \nabla \boldsymbol{\eta}(x), \nabla \mathbf{a}_3(\boldsymbol{\eta}(x)), (1, \varepsilon H(\boldsymbol{\eta}(x), \varepsilon^2 K(\boldsymbol{\eta}(x)) \sqrt{a(\boldsymbol{\eta}(x))})) \\ &= \lim_{n \rightarrow \infty} \mathbb{W}(x, \sum_{t=n}^{j(n)} \mu_t^n (\nabla \boldsymbol{\eta}^t(x), \nabla \mathbf{a}_3(\boldsymbol{\eta}^t(x)), \mathbf{v}(\boldsymbol{\eta}^t(x))) \end{aligned}$$

where  $\mathbf{v}(\boldsymbol{\eta}^t(x)) := (1, \varepsilon H(\boldsymbol{\eta}^t(x)), \varepsilon^2 K(\boldsymbol{\eta}^t(x)) \sqrt{a(\boldsymbol{\eta}^t(x))})$ . Using this relation, Fatou's lemma, and the assumed convexity of the function  $\mathbb{W}(x, \cdot)$  for almost all  $x \in \omega$ , we next obtain, on the one hand,

$$\begin{aligned} \int_{\omega} W(x, \boldsymbol{\eta}) \, dx &\leq \liminf_{n \rightarrow \infty} \int_{\omega} \mathbb{W}\left(x, \sum_{t=n}^{j(n)} \mu_t^n (\nabla \boldsymbol{\eta}^t(x), \nabla \mathbf{a}_3(\boldsymbol{\eta}^t(x)), \mathbf{v}(\boldsymbol{\eta}^t(x)))\right) \, dx \\ &\leq \liminf_{n \rightarrow \infty} \sum_{t=n}^{j(n)} \mu_t^n \int_{\omega} W(x, \boldsymbol{\eta}^t) \, dx = \lim_{n \rightarrow \infty} \int_{\omega} W(x, \boldsymbol{\eta}^n) \, dx \\ &= \lim_{m \rightarrow \infty} \int_{\omega} W(x, \boldsymbol{\eta}^m) \, dx. \end{aligned}$$

Note that we have also used here a simple observation: Let  $(\alpha^n)$  be a convergent sequence of real numbers, and let

$$\beta^n := \sum_{t=n}^{j(n)} \mu_t^n \alpha^t \quad \text{with } \mu_t^n \geq 0 \quad \text{and} \quad \sum_{t=n}^{j(n)} \mu_t^n = 1 \text{ for each } n.$$

Then the sequence  $(\beta^n)$  is also convergent, and  $\lim_{n \rightarrow \infty} \beta^n = \lim_{n \rightarrow \infty} \alpha^n$ .

Since, on the other hand,  $L(\boldsymbol{\eta}, \mathbf{a}_3(\boldsymbol{\eta})) = \lim_{\ell \rightarrow \infty} L(\boldsymbol{\eta}^\ell, \mathbf{a}_3(\boldsymbol{\eta}^\ell))$  by definition of weak convergence, we have therefore proved that

$$I(\boldsymbol{\eta}) \leq \liminf_{\ell \rightarrow \infty} I(\boldsymbol{\eta}^\ell).$$

(vi) *The function  $\boldsymbol{\eta}$  is thus a solution of the minimization problem, since  $\boldsymbol{\eta} \in \mathbf{V}^\varepsilon$  by parts (iii) and (iv), and since*

$$I(\boldsymbol{\eta}) \leq \liminf_{\ell \rightarrow \infty} I(\boldsymbol{\eta}^\ell) = \inf_{\boldsymbol{\psi} \in \mathbf{V}^\varepsilon} I(\boldsymbol{\psi}) \quad \text{implies} \quad I(\boldsymbol{\eta}) = \inf_{\boldsymbol{\psi} \in \mathbf{V}^\varepsilon} I(\boldsymbol{\psi}).$$

□

## 5 Polyconvex stored energy functions

In this section, we define a general class of polyconvex stored energy functions. As a preparation, we will need the following theorem which is due to (Thompson and Freede, 1971). We recall that the singular values  $\nu_i(\mathbf{F})$  of a matrix  $\mathbf{F} \in \mathbb{M}^{m \times n}$  are the square roots of the eigenvalues of the positive semi-definite matrix  $\mathbf{F}^T \mathbf{F}$ .

**Theorem 2.** *Let there be given a function  $\Phi : [0, +\infty[ \rightarrow \mathbb{R}$  that is symmetric, convex, and nondecreasing in each variable. Then the function*

$$W : \mathbf{F} \in \mathbb{M}^n \rightarrow \Phi(\nu_1(\mathbf{F}), \nu_2(\mathbf{F}), \dots, \nu_n(\mathbf{F}))$$

*is convex.*

**Proposition 1.** *Let  $\alpha \geq 1$ . The function*

$$\hat{W} : \mathbf{A} \in \mathbb{M}^{3 \times 2} \rightarrow \text{tr}\{(\mathbf{A}^T \mathbf{A})^{\alpha/2}\}$$

*is convex.*

*Proof.* The particular function

$$\Phi_\alpha(\nu_1, \nu_2, \nu_3) := \nu_1^\alpha + \nu_2^\alpha + \nu_3^\alpha, \quad \alpha \geq 1$$

satisfies all the assumptions of Theorem 2. Hence the function

$$W : \mathbf{F} \in \mathbb{M}^3 \rightarrow \text{tr}\{(\mathbf{F}^T \mathbf{F})^{\alpha/2}\}$$

is convex. Let  $\mathbf{A} \in \mathbb{M}^{3 \times 2}$ . By letting  $\mathbf{F} := (\mathbf{A} | \mathbf{0}) \in \mathbb{M}^3$  where the third column of  $\mathbf{F}$  is  $\mathbf{0}$ , it follows that

$$W(\mathbf{F}) = \hat{W}(\mathbf{A}).$$

Hence,  $\hat{W}$  is convex on  $\mathbb{M}^{3 \times 2}$ . □

Now, we recall the notation used in the following theorem. For  $z \in [-\varepsilon, \varepsilon]$  and for all  $\alpha \in \{1, 2\}$ , we denote

$$\mathbf{g}_\alpha(z) := \partial_\alpha \boldsymbol{\varphi} + z \partial_\alpha \mathbf{a}_3$$

and  $(\mathbf{g}^\alpha(z))$  the corresponding contravariant basis.

**Theorem 3.** *Let  $u, v \in [-\varepsilon, \varepsilon]$ ,  $\boldsymbol{\psi} \in \mathbf{V}^\varepsilon$  and*

$$\begin{aligned} \mathbf{G}^+(\boldsymbol{\psi}) &:= \{a_{\alpha\beta}(\boldsymbol{\psi}) - 2ub_{\alpha\beta}(\boldsymbol{\psi}) + u^2c_{\alpha\beta}(\boldsymbol{\psi})\} \mathbf{g}^\alpha(v) \otimes \mathbf{g}^\beta(v), \\ \mathbf{G}^-(\boldsymbol{\psi}) &:= \{a_{\alpha\beta}(\boldsymbol{\psi}) + 2ub_{\alpha\beta}(\boldsymbol{\psi}) + u^2c_{\alpha\beta}(\boldsymbol{\psi})\} \mathbf{g}^\alpha(v) \otimes \mathbf{g}^\beta(v). \end{aligned}$$

*Then, for all  $\alpha \geq 2$ , there exists a constant  $C > 0$  such that for all  $\boldsymbol{\psi} \in \mathbf{V}^\varepsilon$  and all  $u, v \in [-\varepsilon, \varepsilon]$ ,*

$$\text{tr}\{\mathbf{G}^+(\boldsymbol{\psi})^{\alpha/2} + \mathbf{G}^-(\boldsymbol{\psi})^{\alpha/2}\} \geq C\{|\nabla \boldsymbol{\psi}|^\alpha + |u|^\alpha |\nabla \mathbf{a}_3(\boldsymbol{\psi})|^\alpha\}.$$

*Proof.* Let  $(\mathbf{e}_1, \mathbf{e}_2)$  denote the canonical basis of  $\mathbb{R}^2$  and

$$\mathbf{D} := \mathbf{e}_\alpha \otimes \mathbf{g}^\alpha(v).$$

Then

$$\mathbf{G}^+(\boldsymbol{\psi}) = \mathbf{D}^\top (\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi}))^\top (\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})) \mathbf{D}.$$

Since, for  $\gamma \geq 1$ , the mapping

$$(v_i) \in \mathbb{R}^3 \rightarrow (|v_1|^\gamma + |v_2|^\gamma + |v_3|^\gamma)^{1/\gamma}$$

is a norm on the space  $\mathbb{R}^3$ , and since all norms are equivalent on a finite-dimensional space, there exists for each  $\gamma \geq 1$  a constant  $C_\gamma$  such that

$$|v_1|^\gamma + |v_2|^\gamma + |v_3|^\gamma \geq C_\gamma (|v_1|^2 + |v_2|^2 + |v_3|^2)^{\gamma/2}.$$

We thus have

$$\text{tr}\{\mathbf{G}^+(\boldsymbol{\psi})^{\alpha/2}\} \geq C_\alpha \{\text{tr} \mathbf{G}^+(\boldsymbol{\psi})\}^{\alpha/2} \geq C_\alpha |(\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})) \mathbf{D}|^\alpha.$$

Since  $\mathbf{D} \mathbf{g}_\beta(v) = \mathbf{e}_\beta$ , it follows that

$$\mathbf{D}(\nabla \boldsymbol{\varphi} + v \nabla \mathbf{a}_3) = \mathbf{I}_2,$$

where  $\mathbf{I}_2 \in \mathbb{M}^2$  is the identity matrix. Hence

$$\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi}) = (\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})) \mathbf{D} (\nabla \boldsymbol{\varphi} + v \nabla \mathbf{a}_3).$$

Then

$$\begin{aligned} |\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})| &\leq |(\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})) \mathbf{D}| |\nabla \boldsymbol{\varphi} + v \nabla \mathbf{a}_3| \\ &\leq |(\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})) \mathbf{D}| (\|\nabla \boldsymbol{\varphi}\|_{\infty, \omega} + \varepsilon \|\nabla \mathbf{a}_3\|_{\infty, \omega}). \end{aligned}$$

Therefore, there exists a constant  $C > 0$  such that

$$\text{tr}\{\mathbf{G}^+(\boldsymbol{\psi})^{\alpha/2}\} \geq C |\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})|^\alpha$$

and there exists a constant  $C > 0$  such that

$$\begin{aligned} \text{tr}\{\mathbf{G}^+(\boldsymbol{\psi})^{\alpha/2} + \mathbf{G}^-(\boldsymbol{\psi})^{\alpha/2}\} \\ \geq C \{|\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})|^2 + |\nabla \boldsymbol{\psi} - u \nabla \mathbf{a}_3(\boldsymbol{\psi})|^2\}^{\alpha/2}. \end{aligned}$$

But

$$|\nabla \boldsymbol{\psi} + u \nabla \mathbf{a}_3(\boldsymbol{\psi})|^2 + |\nabla \boldsymbol{\psi} - u \nabla \mathbf{a}_3(\boldsymbol{\psi})|^2 = 2|\nabla \boldsymbol{\psi}|^2 + 2|u|^2 |\nabla \mathbf{a}_3(\boldsymbol{\psi})|^2.$$

It follows that there exists a constant  $C > 0$  such that

$$\text{tr}\{\mathbf{G}^+(\boldsymbol{\psi})^{\alpha/2} + \mathbf{G}^-(\boldsymbol{\psi})^{\alpha/2}\} \geq C\{|\nabla \boldsymbol{\psi}|^\alpha + |u|^\alpha |\nabla \mathbf{a}_3(\boldsymbol{\psi})|^\alpha\}$$

and the proof is complete.  $\square$

**Theorem 4.** Let  $\Gamma : \mathbf{N} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  be a convex function on

$$\mathbf{N} := \{(a, b, c) \in \mathbb{R}^3, a - |b| > 0 \text{ and } a - 2|b| + c > 0\}$$

and let  $W : \omega \times \mathbf{V}^\varepsilon \rightarrow \mathbb{R}$  be a stored energy function defined by

$$\begin{aligned} W(x, \boldsymbol{\psi}) &= \sum_{i=1}^R a_i \text{tr}\{\mathbf{G}_i^+(\boldsymbol{\psi})^{\gamma_i/2} + \mathbf{G}_i^-(\boldsymbol{\psi})^{\gamma_i/2}\} \\ &\quad + \Gamma(\sqrt{a(\boldsymbol{\psi})}, \varepsilon H(\boldsymbol{\psi})\sqrt{a(\boldsymbol{\psi})}, \varepsilon^2 K(\boldsymbol{\psi})\sqrt{a(\boldsymbol{\psi})}) \end{aligned}$$

where

$$\begin{aligned} \mathbf{G}_i^+(\boldsymbol{\psi}) &:= \{a_{\alpha\beta}(\boldsymbol{\psi}) - 2z_i b_{\alpha\beta}(\boldsymbol{\psi}) + (z_i)^2 c_{\alpha\beta}(\boldsymbol{\psi})\} \mathbf{g}^\alpha(z_i) \otimes \mathbf{g}^\beta(z_i), \\ \mathbf{G}_i^-(\boldsymbol{\psi}) &:= \{a_{\alpha\beta}(\boldsymbol{\psi}) + 2z_i b_{\alpha\beta}(\boldsymbol{\psi}) + (z_i)^2 c_{\alpha\beta}(\boldsymbol{\psi})\} \mathbf{g}^\alpha(-z_i) \otimes \mathbf{g}^\beta(-z_i) \end{aligned}$$

and  $a_i > 0$ ,  $\gamma_i \geq 2$ ,  $z_i \in [-\varepsilon, \varepsilon]$ ,  $1 \leq i \leq R$ .

Then the function  $W$  is polyconvex and satisfies a coerciveness inequality of the form

$$\begin{aligned} W(x, \boldsymbol{\psi}) &\geq C\{|\nabla \boldsymbol{\psi}|^{\gamma_{i_0}} + |z_{i_0}|^{\gamma_{i_0}} |\nabla \mathbf{a}_3(\boldsymbol{\psi})|^{\gamma_{i_0}}\} \\ &\quad + \Gamma(\sqrt{a(\boldsymbol{\psi})}, \varepsilon H(\boldsymbol{\psi})\sqrt{a(\boldsymbol{\psi})}, \varepsilon^2 K(\boldsymbol{\psi})\sqrt{a(\boldsymbol{\psi})}) \end{aligned}$$

with a constant  $C > 0$  and  $\gamma_{i_0} = \max_i \gamma_i$ .

*Proof.* (i) Fix  $z_i \in [-\varepsilon, \varepsilon]$ . Let  $(\mathbf{e}_1, \mathbf{e}_2)$  denote the canonical basis of  $\mathbb{R}^2$ ,

$$\mathbf{D}_i^+ := \mathbf{e}_\alpha \otimes \mathbf{g}^\alpha(z_i) \quad \text{and} \quad \mathbf{D}_i^- := \mathbf{e}_\alpha \otimes \mathbf{g}^\alpha(-z_i).$$

Let  $\mathbb{W}_i^+(x, \cdot) : (\mathbf{A}, \mathbf{B}) \in (\mathbb{M}_{3 \times 2})^2 \rightarrow \mathbb{R}$  be defined by

$$\mathbb{W}_i^+(x, \mathbf{A}, \mathbf{B}) := \text{tr}\left\{((\mathbf{D}_i^+)^T(\mathbf{A} + z_i \mathbf{B})^T(\mathbf{A} + z_i \mathbf{B})\mathbf{D}_i^+)^{\gamma_i/2}\right\}.$$

As a composition of the linear function

$$(\mathbf{A}, \mathbf{B}) \in (\mathbb{M}_{3 \times 2})^2 \rightarrow \{\mathbf{A} + z_i \mathbf{B}\} \mathbf{D}_i^+$$

and the function

$$\mathbf{G} \in \mathbb{M}_3 \rightarrow \text{tr}\{\mathbf{G}^T \mathbf{G}\}^{\gamma_i/2}$$

which is convex by Proposition 1, it follows that the function  $\mathbb{W}_i^+(x, \cdot)$  is convex. Besides, by using the same argument, we prove that the function  $\mathbb{W}_i^-(x, \cdot) : (\mathbf{A}, \mathbf{B}) \in (\mathbb{M}_{3 \times 2})^2 \rightarrow \mathbb{R}$  defined by

$$\mathbb{W}_i^-(x, \mathbf{A}, \mathbf{B}) := \text{tr}\{(\mathbf{D}_i^-)^T \{\mathbf{A} - z_i \mathbf{B}\}^T \{\mathbf{A} - z_i \mathbf{B}\} \mathbf{D}_i^-\}^{\gamma_i/2}$$

is convex. Finally, let

$$\mathbf{M} := \{(\mathbf{A}, \mathbf{B}, a, b, c) \in (\mathbb{M}_{3 \times 2})^2 \times \mathbb{R}^3, (a, b, c) \in \mathbf{N}\}.$$

The function  $\mathbb{W}(x, \cdot) : (\mathbf{A}, \mathbf{B}, a, b, c) \in \mathbf{M} \rightarrow \mathbb{R}$  defined by

$$\mathbb{W}(x, \mathbf{A}, \mathbf{B}, a, b, c) := \sum_{i=1}^M a_i \{\mathbb{W}_i^+(x, \mathbf{A}, \mathbf{B}) + \mathbb{W}_i^-(x, \mathbf{A}, \mathbf{B})\} + \Gamma(a, b, c)$$

is convex. Since

$$\begin{aligned} \text{tr}\{\mathbf{G}_i^+(\boldsymbol{\psi})^{\gamma_i/2}\} &= \mathbb{W}_i^+(x, \nabla \boldsymbol{\psi}, \nabla \mathbf{a}_3(\boldsymbol{\psi})) \\ \text{and } \text{tr}\{\mathbf{G}_i^-(\boldsymbol{\psi})^{\gamma_i/2}\} &= \mathbb{W}_i^-(x, \nabla \boldsymbol{\psi}, \nabla \mathbf{a}_3(\boldsymbol{\psi})), \end{aligned}$$

it follows that

$$W(x, \boldsymbol{\psi}) = \mathbb{W}(x, \nabla \boldsymbol{\psi}, \nabla \mathbf{a}_3(\boldsymbol{\psi}), \sqrt{a(\boldsymbol{\psi})}, \varepsilon H(\boldsymbol{\psi}) \sqrt{a(\boldsymbol{\psi})}, \varepsilon^2 K(\boldsymbol{\psi}) \sqrt{a(\boldsymbol{\psi})}).$$

Thus the function  $W$  is polyconvex.

(ii) The coerciveness inequality follows directly by applying the Theorem 3.  $\square$

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